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SEPARABILITY, EXTERNALITIES, AND
COMPETITIVE EQUILIBRIUM

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1. INTRODUCTION

The characterization of external effects as "separable" has played an important role in the development of the theory of externalities. The separable case appears particularly well behaved when procedures for achieving an optimum allocation of resources in the presence of externalities are examined. Three examples can illustrate the range of conclusions which have been reached concerning separable externalities. Davis and Whinston [1962] find that separability assures the existence of a certain kind of equilibrium in bargaining between firms which create externalities, and that equilibrium does not exist without separability. Kneese and Bower [1968] argue that with separability the computation of Pigovian taxes to remedy externalities is particularly simple. Marchand and Russell [1974] demonstrate that certain liability rules regarding external effects lead to Pareto optimal outcomes if and only if externalities are separable. In these and other articles an externality is defined as separable if the cost function of an affected firm has a specific form, stated in Definition 1 of this paper. With few exceptions, explorations of the implications of separability have assumed that equilibria and optima can be characterized in terms of classical first-order conditions of profit-maximization. Examination of the class of production functions

which are compatible with separable externalities reveals, however, that separable externalities cause a distinctive non-convexity when the possibility that a firm will shut down rather than accept negative profits is introduced. Since numerous policies for dealing, for example, with environmental damage have been based on theoretical investigations of externalities, a defect in those investigations can have serious consequences. In this paper we characterize the class of production functions which generate separable externalities. These results are used to show that all production functions in this class contain a non-convex part. Some of the consequences of this non-convexity for market structure in the presence of separable externalities are examined. Finally, examples are given which suggest some conditions under which a competitive equilibrium may exist in the presence of externalities and some conditions under which it may not.

2. SEPARABLE EXTERNALITIES IN COST AND PRODUCTION FUNCTIONS

Let $C(Y_1, Y_2)$ be the cost function of a firm which produces Y_1 , and suffers an external diseconomy which is a function of Y_2 . That cost function is defined in terms of a production function $F(X_1, \dots, X_n, Y_2)$, in the following manner:

$$C(Y_1, Y_2) = \min \sum_i W_i X_i$$

subject to $Y_1 = F(X_1, \dots, X_n, Y_2)$.

We assume throughout that C and F are continuously twice differentiable, and that F is strictly quasi-concave. Some precise definitions and lemmas regarding separability are needed:¹

¹These strong definitions are defended and related to alternative definitions of separability in an Appendix to this paper.

Definition 1: A cost function $C(Y_1, Y_2)$ is separable if and only if it can be written as $C_1(Y_1) + C_2(Y_2)$.

Definition 2: A production function $F(X_1, \dots, X_n, Y_2)$ is separable if and only if it can be written as $g(X_1, \dots, X_n) + h(Y_2)$.

Lemma 1: A cost function is separable if and only if $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$ everywhere.

Proof: Necessity is proved by differentiating $C = C_1 + C_2$ twice and observing that $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$. Sufficiency is proved² by observing that the general solution of the second-order partial differential equation $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$ is of the form $C = C_1(Y_1) + C_2(Y_2)$.

²Lester Ford, [1955], p. 251 derives this result.

The problem is to find what general class of functions F give rise to cost functions with the property $\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$. It has been claimed by Marchand and Russell that a cost function is separable if and only if the production function from which it is derived is separable as defined above. This conjecture is false in both directions. We begin by giving a simple example of a separable production function which does not generate a separable cost function. It will be seen later that the class of separable functions does not exhaust the class of functions giving rise to separable cost functions.

Consider a production function of the form

$$F = X_1^\alpha X_2^\beta - CY_2 \quad (1)$$

where $\alpha + \beta < 1$. We find the cost function by solving the cost-minimization problem and using the first-order conditions and the production function to eliminate the inputs from the cost equation. From the first-order conditions we have

$$\frac{W_1}{W_2} = \frac{\alpha X_2}{\beta X_1} \quad (2)$$

Solving for X_1 and substituting in (1) gives

$$Y_1 = X_2^{\alpha+\beta} \left(\frac{W_2^\alpha}{W_1^\beta} \right)^\alpha - CY_2 \quad (3)$$

Solving (3) for X_2 , and substituting the resulting expression for X_2 in (2) enables us to express X_1 and X_2 in terms of Y_1 and Y_2 alone. Substitution in $C = W_1 X_1 + W_2 X_2$ gives the cost function

$$C = W_2 \left(1 + \frac{\alpha}{\beta} \right) \left(\frac{W_2^\alpha}{W_1^\beta} \right)^{\frac{\alpha}{\alpha+\beta}} (Y_1 + CY_2)^{\frac{1}{\alpha+\beta}} \quad (4)$$

Clearly (4) is not separable if $\alpha + \beta \neq 1$.

To find a production function which does generate a separable cost function we express $\frac{\partial^2 C}{\partial Y_1 \partial Y_2}$ in terms of the derivatives of the production function, and then find solutions of the partial differential equation which results when $\frac{\partial^2 C}{\partial Y_1 \partial Y_2}$ is set equal to zero.

The general relation between cost and production functions is found by adopting the approach of Samuelson's Foundations. Consider the constrained cost minimization problem

$$\text{Minimize } \sum_i W_i X_i \text{ subject to } Y_1 - F(X_1, \dots, X_n) = 0.$$

We adopt the following abbreviations

$$\begin{aligned} \frac{\partial F}{\partial X_i} &= F_i & \frac{\partial F}{\partial Y} &= F_Y \\ \frac{\partial^2 F}{\partial X_i \partial X_j} &= F_{ij} & \frac{\partial^2 F}{\partial X_i \partial Y} &= F_{iY} \end{aligned}$$

$$\text{Let } \Delta = \begin{vmatrix} F_{11} & \dots & F_{1n} & F_1 \\ \vdots & & \vdots & \vdots \\ F_{n1} & \dots & F_{nn} & F_n \\ F_1 & \dots & F_n & 0 \end{vmatrix}$$

Further let Δ_{ij} be the i, j th co-factor of Δ

Theorem 1: A cost function is separable if and only if it is derived from a production function which satisfies

$$\sum_i F_{iY} \Delta_{n+1,i} + \Delta_{n+1,n+1} F_Y = 0$$

at every point which is a proper cost minimum.

Proof: By Lemma 1 the cost function is separable if and only if

$\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0$. We express $\frac{\partial^2 C}{\partial Y_1 \partial Y_2}$ in terms of the production function as follows.

Form the Lagrangian expression

$$L = \sum_i W_i X_i + \lambda (Y_1 - F(X_1, \dots, X_n, Y_2))$$

First-order conditions are

$$W_i - \lambda F_i = 0$$

$$Y_1 - F = 0$$

We perturb the solution by varying Y_1 and Y_2 . Totally differentiating the first-order conditions gives the system of equations

$$\begin{pmatrix} F_{11} & \dots & F_{1n} & F_1 \\ \vdots & & \vdots & \vdots \\ F_{n1} & \dots & F_{nn} & F_n \\ F_1 & \dots & F_n & 0 \end{pmatrix} \begin{pmatrix} dX_1 \\ \vdots \\ dX_n \\ d\lambda/\lambda \end{pmatrix} = \begin{pmatrix} \frac{dW_1}{\lambda} - F_{1Y} dY_2 \\ \vdots \\ \frac{dW_n}{\lambda} - F_{nY} dY_2 \\ dY_1 - F_Y dY_2 \end{pmatrix} \quad (6)$$

Solving for dX_k using Cramer's rule gives

$$dX_k = \frac{\sum_{i=1}^n \left[\left(\frac{dW_i}{\lambda} - F_{iY} dY_2 \right) \Delta_{ik} + \left(dY_1 - F_Y^1 dY_2 \right) \Delta_{n+1,k} \right]}{\Delta}$$

We assume that F is strictly quasi-concave in $X_1 \dots X_n$, so that $\Delta \neq 0$.

Then

$$\frac{\partial X_k}{\partial Y_2} = \frac{-\sum_i F_{iY} \Delta_{ik} - F_Y \Delta_{n+1,k}}{\Delta}$$

Since $C = \sum_k W_k X_k$,

$$\frac{\partial C}{\partial Y_2} = \sum_k W_k \frac{\partial X_k}{\partial Y_2}$$

and

$$\frac{\partial C}{\partial Y_2} = \frac{-\sum_k W_k \left[\sum_i F_{iY} \Delta_{ik} + F_Y \Delta_{n+1,k} \right]}{\Delta}$$

Since $W_k = \lambda F_k$,

$$\frac{\partial C}{\partial Y_2} = \frac{-\sum_{i=1}^n \left[F_{iY} \lambda \left(\sum_k F_k \Delta_{ik} \right) \right] - \lambda F_Y \sum_k F_k \Delta_{n+1,k}}{\Delta}$$

But $\sum_k F_k \Delta_{n+1,k} = \Delta$, and $\sum_k F_k \Delta_{ik} = 0$ since it is an expansion by alien co-factors. Therefore

$$\frac{\partial C}{\partial Y_2} = -\lambda F_Y \quad (7)$$

Differentiating (7) with respect to Y_1 gives

$$\frac{\partial^2 C}{\partial Y_2 \partial Y_1} = -\lambda \sum_i F_{iY} \frac{\partial X_i}{\partial Y_1} - F_Y \frac{\partial \lambda}{\partial Y_1}$$

From (6),

$$\frac{\partial \lambda}{\partial Y_1} = \lambda \frac{\Delta_{n+1,n+1}}{\Delta}, \quad \frac{\partial X_i}{\partial Y_1} = \frac{\Delta_{n+1,i}}{\Delta}$$

Therefore

$$\frac{\partial^2 C}{\partial Y_2 \partial Y_1} = -\frac{\lambda}{\Delta} \left(\sum_i F_{iY} \Delta_{n+1,i} + \Delta_{n+1,n+1} F_Y \right)$$

Characterizing the class of production functions which generate separable cost functions reduces to finding the general form of the solution of

$$\sum_i F_{iY} \Delta_{n+1,i} + \Delta_{n+1,n+1} F_Y = 0 \quad (8)$$

Three immediate consequences of (8) are of interest.

Corollary 1: If F is of the form of $g(X_1, \dots, X_n) + h(Y_2)$ where g is homogeneous of degree one, then F generates a separable cost function.

Proof: Obviously $F_{iY} = 0$ for all i . If g is homogeneous of degree one then $|g_{ij}| = \Delta_{n+1, n+1} = 0$ wherever evaluated [Quirk and Saposnik, 1963]. Therefore F satisfies (8).

Corollary 2: If the cost function $C(Y_1, Y_2)$ and the production function $F(X_1, \dots, X_n, Y_2)$ are both separable and $F_Y \neq 0$ everywhere, then $|g_{ij}| = 0$.

Proof: By separability of the cost function (8) holds. Separability of the production function implies $F_{iY} = 0$ for all i . Therefore

$$\Delta_{n+1, n+1} = |g_{ij}| = 0$$

Corollary 3: If the production function $F(X_1, \dots, X_n, Y_2)$ is separable, $F_Y \neq 0$ and $|g_{ij}| \neq 0$, then the cost function is not separable.

Proof: By hypothesis $F_{iY} = 0$ and $F_Y \Delta_{n+1, n+1} \neq 0$. Therefore (8) does not vanish.

It is possible to find a general solution for (8) when there is just one input, denoted X . Then (8) becomes

$$F_X F_{YX} - F_Y F_{XX} = 0 \quad (9)$$

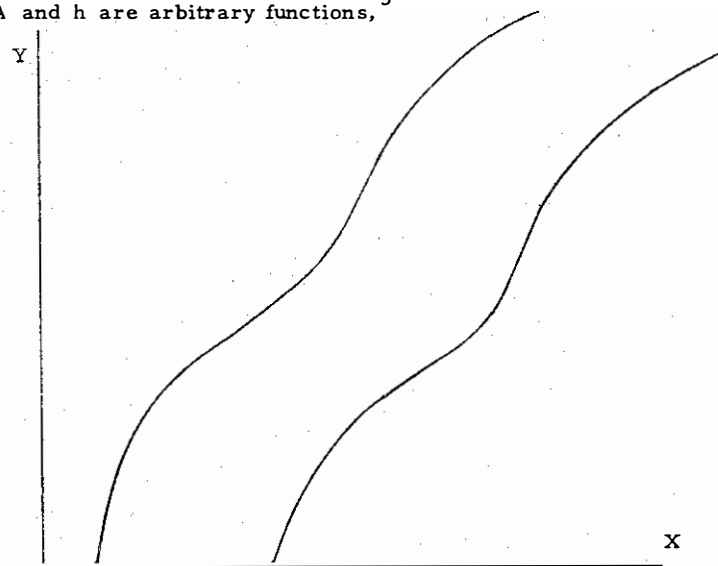
(9) is equal to the numerator of the expression $\frac{\partial}{\partial X} \left(\frac{F_Y}{F_X} \right)$. Thus the solution of (9) will be a function such that the ratio of F_Y to F_X is independent of X .

$$\text{Let } \frac{F_Y}{F_X} = -\phi(Y)$$

Then for any fixed value of F ,

$$\frac{dX}{dY} = \phi(Y)$$

In Figure 1, these level surfaces are illustrated. Each is simply a horizontal displacement of some other level surface (isoquant). Denote each isoquant $C(\lambda)$. Then on any isoquant $X = \lambda + f(Y)$ where f is some arbitrary function. The function F which solves (9) is an arbitrary function of λ , say $F = A(\lambda)$. Thus $F = A(X + h(Y))$, where A and h are arbitrary functions,³



which may be restricted to preserve the concavity and strict quasi-concavity of F . We check that this solution works by differentiating:

$$F_X = A'$$

$$F_Y = h' A'$$

$$\frac{F_Y}{F_X} = h'(Y)$$

³ I am indebted to Joel Franklin for this demonstration.

There are two obvious ways of generalizing the form $F = A(X + h(Y_2))$ to the n -input case:

$$F = B(X_1 + h_1(Y_2), \dots, X_n + h_n(Y_2)) \quad (10)$$

where B and h_1, \dots, h_n are arbitrary functions or

$$F = A(g(X_1, \dots, X_n) + h(Y_2)) \quad (11)$$

where A and h are arbitrary functions and $|g_{ij}| = 0$. We will show that both (10) and (11) are solutions of (8). Though (10) and (11) bear some obvious resemblances to each other, it does not appear that either is the most general form of a solution of (8).

The proof that (10) solves (8) is nearly trivial. We compute

$$\frac{\partial^2 F}{\partial X_i \partial Y_2} = \sum_j B_{ij} h'_j$$

and

$$\frac{\partial F}{\partial Y_2} = \sum_i B_i h'_i$$

Substituting, (8) becomes

$$\sum_j h'_j \left(\sum_i B_{ij} \Delta_{n+1, i} + \Delta_{n+1, n+1} B_j \right)$$

This expression is identically zero, since the expression in parentheses is an expansion of Δ by alien co-factors. (The j^{th} column of the matrix is multiplied by the co-factors of the $n+1$ column.)

Somewhat more work is needed to show that (8) is satisfied when F has the form

$$A(g(X_1, \dots, X_n) + h(Y_2))$$

where A and h are arbitrary functions and $|g_{ij}| = 0$. We demonstrate that this is the case by establishing a relation between co-factors of

$$\begin{vmatrix} A_{ij} & A_i \\ A_j & 0 \end{vmatrix} = |A|$$

and co-factors of

$$\begin{vmatrix} g_{ij} & g_i \\ g_j & 0 \end{vmatrix} = |G|$$

It is well known that $|A| = (A')^{n+1} |G|$.

Write

$$\begin{pmatrix} A_{ij} & A_i \\ A_i & 0 \end{pmatrix} = \begin{pmatrix} A''_1 g_1^2 + A'_1 g_{11} & \dots & A''_1 g_1 g_n + A'_1 g_{1n} & A'_1 g_1 \\ \vdots & & \vdots & \vdots \\ A''_n g_n g_1 + A'_n g_{n1} & \dots & A''_n g_n^2 + A'_n g_{nn} & A'_n g_n \\ A'_1 g_1 & \dots & A'_n g_n & 0 \end{pmatrix}$$

Then (8) becomes

$$\sum_i A'' h' g_i |A|_{n+1,i} + A' h' |A|_{n+1,n+1} = 0$$

Since $\sum_i A' g_i |A|_{n+1,i} = |A|$, we have, if $h' \neq 0$

$$\frac{A''}{A} |A| + A' |A|_{n+1,n+1} = 0 \quad (12)$$

Lemma 3: If

$$C = \begin{pmatrix} C_{ij} & \vdots & C_i \\ \vdots & \ddots & \vdots \\ C_j & \vdots & 0 \end{pmatrix}$$

then

$$|C| = -\sum_i \sum_j C_i C_j |c_{ij}|_{i,j}$$

where $|c_{ij}|_{i,j}$ is the co-factor of C_{ij} in the co-factor $|C|_{n+1,n+1}$.

Proof: Expand $|C|$ by its last column, obtaining

$$|C| = \sum_i C_i |c|_{i,n+1}$$

$$= C_1 \begin{vmatrix} C_{21} & \cdots & C_{2n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{vmatrix} (-1)^{n+2} + \cdots + C_n \begin{vmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n-1,1} & \cdots & C_{n-1,n} \end{vmatrix} (-1)^{2n+1}$$

Now expand each determinant in this sum by its last row, obtaining

$$\begin{aligned} |C| &= C_1 C_1 \begin{vmatrix} C_{22} & \cdots & C_{2n} \\ \vdots & & \vdots \\ C_{n2} & \cdots & C_{nn} \end{vmatrix} (-1)^{n+2} (-1)^{n+1} + \cdots + C_1 C_n \begin{vmatrix} C_{21} & \cdots & C_{2,n-1} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{n,n-1} \end{vmatrix} (-1)^{n+2} (-1)^{2n} \\ &+ C_n C_1 \begin{vmatrix} C_{12} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n-1,2} & \cdots & C_{n-1,n} \end{vmatrix} (-1)^{2n+1} (-1)^{n+1} + \cdots + C_n C_n \begin{vmatrix} C_{11} & \cdots & C_{1,n-1} \\ \vdots & & \vdots \\ C_{n-1,1} & \cdots & C_{n-1,n-1} \end{vmatrix} (-1)^{2n+1} (-1)^{2n} \\ &= \sum_i \sum_j C_i C_j \begin{vmatrix} C_{11} & \cdots & C_{1,j-1}, C_{1,j+1}, \cdots, C_{1n} \\ \vdots & & \vdots & & \vdots \\ C_{i-1,1} & & & & C_{i-1,n} \\ C_{i+1,1} & & & & C_{i+1,n} \\ \vdots & & & & \vdots \\ C_n & \cdots & \cdots & \cdots & C_{nn} \end{vmatrix} (-1)^{2n+1+i+j} \end{aligned}$$

Now

$$|c_{ij}|_{i,j} = (-1)^{i+j} \begin{vmatrix} C_{11} & \cdots & C_{1,j-1}, C_{1,j+1}, \cdots, C_{1n} \\ \vdots & & \vdots & & \vdots \\ C_{i-1,1} & & & & C_{i-1,n} \\ C_{i+1,1} & & & & C_{i+1,n} \\ \vdots & & & & \vdots \\ C_{n,1} & \cdots & C_{n,j-1}, C_{n,j+1}, \cdots, C_{nn} \end{vmatrix}$$

Therefore

$$|C| = -\sum_i \sum_j C_i C_j |c_{ij}|_{i,j}$$

Quirk and Ruppert [1968] have shown that for a matrix of the form of A, using the rule for evaluating the determinant of the sum of two matrices gives

$$|A|_{n+1,n+1} = (A')^n |G|_{n+1,n+1} + A''(A')^{n-1} \sum_i \sum_j g_i g_j |G_{ij}|_{i,j}$$

Using our lemma,

$$\begin{aligned} |A|_{n+1,n+1} &= (A')^n |G|_{n+1,n+1} + A''(A')^{n-1} \sum_i \sum_j g_i g_j |G_{ij}|_{i,j} \\ &= (A')^n |G|_{n+1,n+1} - A''(A')^{n-1} |G| \\ &= (A')^n |G|_{n+1,n+1} - A''(A')^{-2} |A| \end{aligned}$$

Therefore (12) becomes

$$\frac{A''}{A'} |A| + (A')^{n+1} |G|_{n+1,n+1} - \frac{A''}{A'} |A| = 0. \quad (13)$$

But $|G|_{n+1,n+1} = 0$ by hypothesis, so that the left hand side of (13) vanishes identically.

The generalization to the case where firm 1 is affected by externalities from many firms is immediate. We define separability with many sources of externality as follows:

Definition 1': A cost function $C(Y_1, Y_2, \dots, Y_m)$ is separable if and only if it can be written as $C_1(Y_1) + C_2(Y_2, \dots, Y_m)$.

Definition 2': A production function $F(X_1, \dots, X_n, Y_2, \dots, Y_m)$ is separable if and only if it can be written as $g(X_1, \dots, X_n) + h(Y_2, \dots, Y_m)$.

A revised version of Lemma 1 is needed.

Lemma 1': A cost function is separable if and only if everywhere

$$\frac{\partial^2 C}{\partial Y_1 \partial Y_j} = 0 \quad \text{for } j = 2, \dots, m$$

It follows that (8), rewritten as

$$\sum_i F_{iY_j} \Delta_{n+1,i} + \Delta_{n+1,n+1} F_{Y_j} \quad (8')$$

must hold for $j = 2, \dots, m$. An argument almost identical to that used for the case of one firm causing externality establishes that (8') is satisfied by

$$F = A \left(g(X_1, \dots, X_n) + h(Y_2, \dots, Y_m) \right) \quad \text{with } |g_{ij}| = 0$$

$$\text{and by } F = B \left(X_1 + \sum_j h_{1j}(Y_j), \dots, X_n + \sum_j h_{nj}(Y_j) \right).$$

3. SEPARABLE EXTERNALITIES AND COMPETITIVE EQUILIBRIUM

Theorem 1 and its three corollaries establish that representation of externalities as separable in the cost function is not in general equivalent to representation as separable in the production function. Using the derivation of the general form of the one-input production function which gives rise to a separable cost function, we can contrast the behavioral implications of the two forms of separability. In this discussion firm 1 will always be affected by the externality, and firm 2 will always cause the externality. Initially, it is assumed that firm 1 is always at an interior profit maximum, producing positive output from positive input and earning non-negative profits. For reasons to be

discussed shortly, this assumption is not always valid, and the analysis is quite different when it fails.

When the production function is separable, the marginal productivity of any factor used by firm 1 is independent of Y_2 , the output of firm 2. Therefore at an interior maximum for which the first order conditions are necessary, firm 1's input choice is independent of the activity of firm 2. When the cost function is separable, marginal cost of producing Y_1 is independent of Y_2 , and firm 1's output choice is independent of firm 2's activity. If both the production function and the cost function are separable, then input and output of firm 1 are independent of firm 2's activity.⁴

⁴These propositions correct the claim by Wellisz [1964] that separable externalities do not affect resource allocation. It is true that the cost-function separability which he considers implies that the allocation of output is unaltered, but in general the allocation of inputs is affected by devices which "internalize" external effects.

These propositions can be illustrated diagrammatically in a simple fashion for the one-input case. In each figure the production function satisfies $F(0, Y_2) = 0$; i.e., it goes through the origin. In Figure 1 the production function is characterized by decreasing returns and separability. Since the same input must be profit-maximizing for all Y_2 , changes in Y_2 shift the production function vertically, keeping the slope of $F(X_1, Y_2)$ the same for constant X_1 .

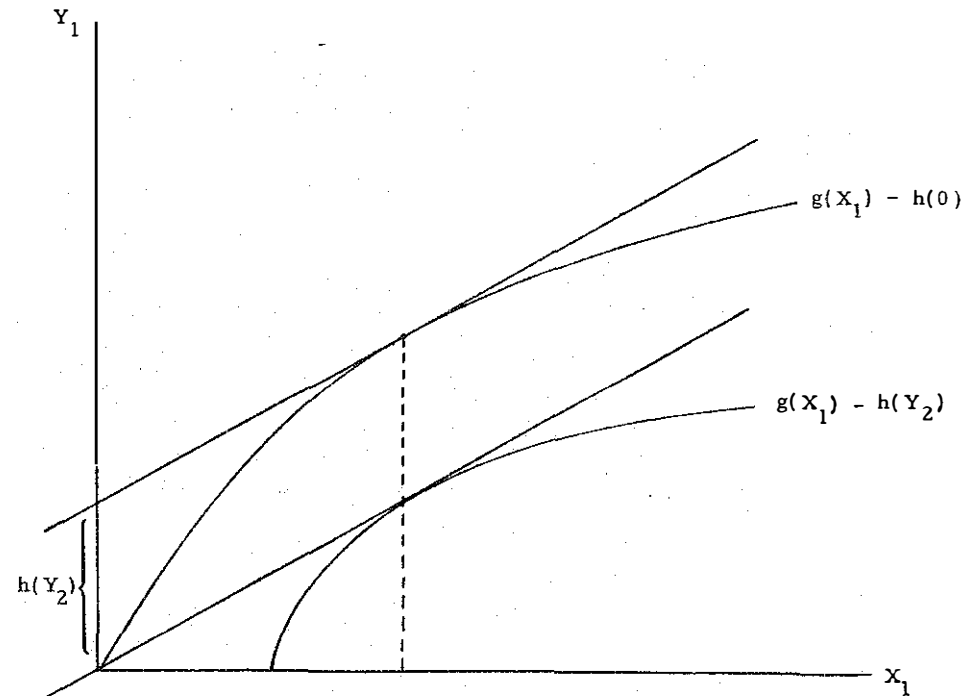


Figure 1

In Figure 2 a production function of the form $F(X_1 - h(Y_2))$ is drawn. Changing Y_2 in this case shifts the production function horizontally, so that the slope of F is constant for constant Y_1 . This production function generates a separable cost function.

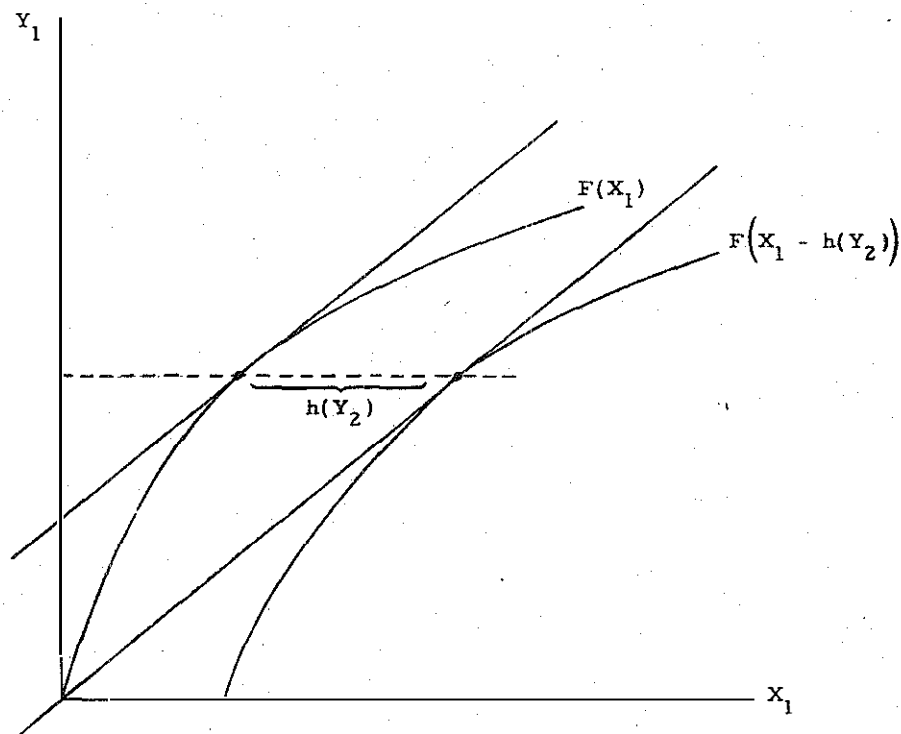


Figure 2

In Figure 3 the production function is separable and generates a separable cost function. Such a function must have the property that the slope of $F(X_1, Y_2)$ equals the slope of $F(X_1, Y_2')$ for constant X_1 and also for constant Y_1 . For this to be true for all X_1 and Y_1 , F must be linear in X_1 , as is proved by Corollary 2.

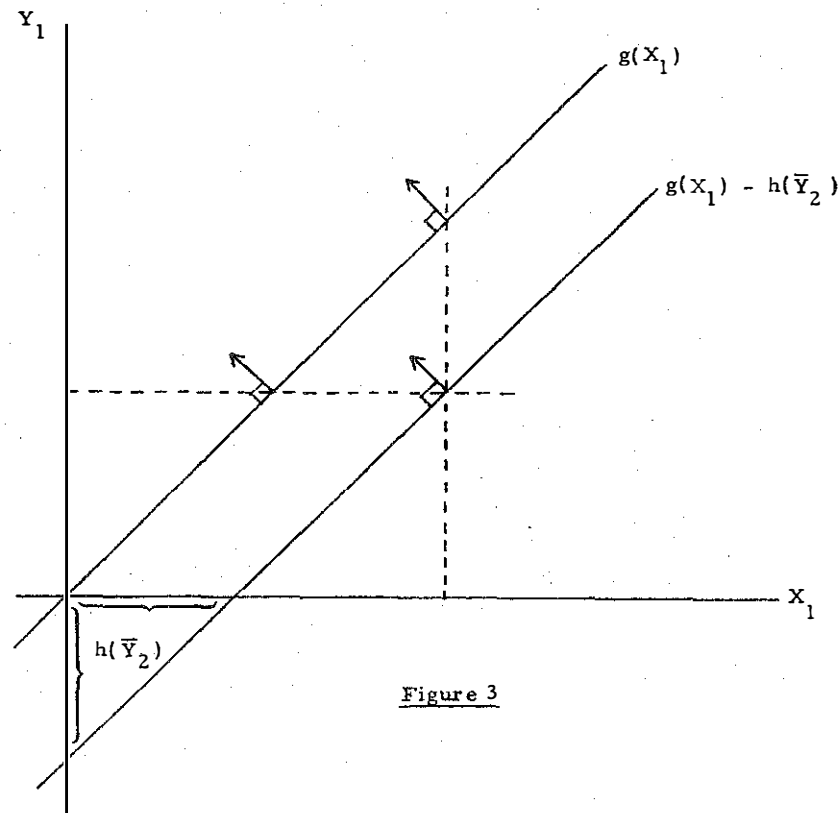


Figure 3

With one input it is only possible to have a separable cost function and a separable production function when the production function is linear in X_1 . But this case is not one which will ever be observed in a market which is in equilibrium, because of a non-convexity which is introduced by the separable externality. Indeed, the same non-convexity besets all situations in which there are separable externalities of any type.⁵

⁵Wellisz, Starrett and Inada and Kuga have also noted the presence of such non-convexity with externalities. The relation of this section to their contribution can be sketched briefly. Starrett and Inada and Kuga analyze a non-convexity which arises for similar reasons to those for which the non-convexity in this paper arises. They consider the implications of the non-convexity for the possibility of achieving a Pareto-optimum when market institutions are used to internalize externalities. We concentrate on the initial competitive equilibrium which is not in general Pareto-optimal, because no markets exist in which externalities could be traded. Wellisz's observations are more in the vein of this paper, since he points out that firms have the choice of shutting down and that separable externalities impose fixed costs.

Consider first the case illustrated in Figure 3. Although these arguments generalize easily to the case in which the production function is separable and $g(X_1, \dots, X_n)$ is homogeneous of degree one, exposition is simplified by considering the one-input case in which $g(X_1)$ is linear and homogeneous. Under these assumptions the production function is a straight line through the origin when $h(Y_2) = 0$. Let $h(0) = 0$ and $h' > 0$. Then for any $Y_2 \neq 0$, firm 1 has a fixed cost inflicted on it by the externality. If the ratio of the price of output to the price of input, p/w , is less than or equal to the slope of the production function, firm 1 will earn negative profits whenever it uses positive input. If p/w is greater than the slope of the production function, firm 1 can earn unbounded positive profits by increasing X_1 without bound. Therefore, if $Y_2 \neq 0$ it will be impossible to observe a firm affected by an externality

which is separable in the cost function and the production function producing finite, non-zero output in equilibrium. This fact has important consequences for the structure of a competitive economy with separable externalities. To explore these consequences we must go from the partial equilibrium analysis used thus far to an explicit general equilibrium approach.

Consider an economy in which there are two firms, each having a production function $g_i(X_i) + h_i(Y_j)$ where $h_i' \neq 0$ everywhere. We can summarize the results of the previous discussion for the case g_i linear and homogeneous by describing the supply correspondence of firm i for fixed Y_j . If $Y_j = 0$ the supply correspondence has the usual properties with constant returns. Let A be the ratio of the price of Y_i to the price of X_i at which firm i can just cover variable costs. It is represented as the slope of a vector orthogonal to the production function. For $p_i/w_i < A$ firm i produces zero; for $p_i/w_i = A$ firm i will produce any output; for $p_i/w_i > A$ supply is undefined. (See Figure 4)

When $Y_j > 0$, the supply correspondence of firm i is not continuous. For price ratios less than or equal to A , $Y_i = 0$. For price ratios greater than A , Y_i is undefined.⁶

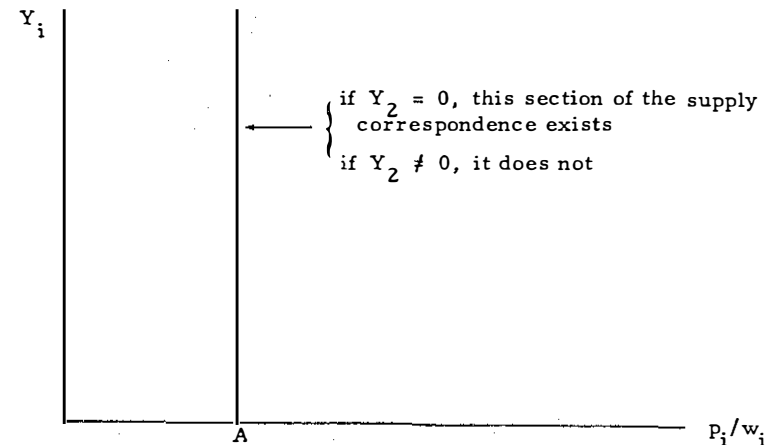


Figure 4

⁶ Note that the supply correspondence has this property whether firm i uses one or many inputs. As long as $g_i(X_i)$ is homogeneous of degree one we can divide the price space into two linear subspaces P_I and P_{II} such that if the price vector $p \in P_I$, the firm supplies zero output, and if $p \in P_{II}$ the supply is undefined. [See Arrow and Hahn, pp. 52-59.]

Three cases can be distinguished: 1) both firms have identical production functions; 2) both firms have identical inputs and outputs; 3) the firms produce different goods, using either the same or different inputs.

Case 1: We ask if there is a price ratio p/w such that when both firms maximize profits $Y_i > 0$ for both firms. If $p/w \leq A$, then $Y_1 > 0$ is not profit-maximizing for firm 1 if $Y_2 > 0$, and vice versa. If $p/w > A$ supply is undefined for both firms. Thus both firms cannot produce positive output in equilibrium. If $Y_1 > 0$ and $Y_2 = 0$, then if $p/w = A$ any finite level of output gives firm 1 zero profits and is in equilibrium. Given that $Y_1 > 0$, $Y_2 = 0$ gives firm 2 maximum profits of zero also.

In this case we can have multiple equilibria, each with just one firm producing non-zero output. It does not matter which firm is out of business. This is characteristic of constant returns, since one firm can produce any level of output using the same total input which would be used if the output were divided among many firms. Normally constant returns imply that the number of firms in an industry is indeterminate. With constant returns and separable externality, the size of an industry is one firm.

Case 2: When the two production functions differ, it will always be the case that with zero externality one firm can always produce more output with given input than could the other. Let A be the slope of a

vector normal to the steepest production function. In equilibrium only the firm with the steepest production function will be in business. The other firm could earn non-negative profits only if $p/w > A$. But then the most productive firm could earn unbounded profits, even when it is suffering externalities from the first firm.

Case 3: If externalities are between firms in different industries, in equilibrium still only one firm produces non-zero output. Suppose two firms are in operation; then if one is affected by externality, it will produce zero or unbounded output depending on the price ratio of its output to its input. Thus at least one firm is not maximizing profit when it chooses non-zero finite output. Again multiple equilibria are possible -- one firm, in any industry, can survive.

The arguments given for two firms also apply to the case of n firms. If it is impossible to have two firms simultaneously maximizing profits and producing positive output, it is impossible to have n , since comparing any two firms, we find only one is in operation. Thus we have proved the following theorem.

Theorem 2: If 1) all externalities are production function separable; 2) g_i is homogeneous of degree one; 3) $h_i \neq 0$ in the neighborhood of equilibrium, then in equilibrium only one firm produces non-zero output.

With constant returns and separability the competitive equilibrium, with only one firm, is also a Pareto optimum. Constant returns imply that in the absence of externality one firm can produce any output as efficiently as many. The fixed cost imposed by externality implies that when more than one firm is in operation more input is needed to produce a given output than is needed when only one firm operates. This conclusion is hardly grounds for optimism, however, since a one-firm market is

unlikely to be competitive. Of course, with separability and decreasing returns the competitive equilibrium will not be optimal for the usual reasons.

4. COMPETITIVE EQUILIBRIUM WITHOUT CONSTANT RETURNS

Standard proofs of the existence of equilibrium of a competitive economy employ assumptions regarding the convexity of production sets which are violated when separable externalities are present. Arrow and Hahn [1972] for example, assume in proving existence with externalities that the production set of firm i is convex in the variables controlled by firm i for every activity chosen by other firms. For the case of separability with constant returns we have given a constructive argument that in equilibrium only one firm will exist. Since with just one firm in operation the fixed cost imposed by externality vanishes, equilibrium itself will exist.

When either the cost function or the production function is separable, and the production function exhibits decreasing returns, a non-convexity will still be present. Note that in both Figures 1 and 2 a fixed cost is imposed on firm 1 when $Y_2 \neq 0$. For some prices this fixed cost will cause firm 1 to shut down. Consider, for example, Figure 2 in which with the price line drawn, firm 1 earns at most zero profits when $Y_2 = \bar{Y}_2$. For any price ratio less than this, firm 1 will produce zero output. When the price ratio is equal to this break-even ratio, firm 1 will jump to a positive level of output, and from this point on its supply function will be continuous. This discontinuity is characteristic of the supply function derived from a production function with a non-convex section. It is generated by any type of separable externality.

To illustrate the implications of this discontinuity for the existence of equilibrium, consider a case in which of the two firms

only one is affected significantly by the externality. That is, assume

$$F_1 = g_1(X_1) - h_1(Y_2)$$

and

$$F_2 = g_2(X_2)$$

Assume for simplicity that both firms produce identical goods, and use identical inputs. Then aggregate supply by these firms is $Y_1 + Y_2$ and aggregate input demand is $X_1 + X_2$. With externality going only one way it is not difficult to determine how $Y_1 + Y_2$ will vary with p/w . At any price ratio firm 2 will set

$$\frac{\partial F_2}{\partial X_2} = \frac{w}{p}$$

The output thus determined is taken as a parameter by firm 1 in choosing its optimal supply. We take a simple example:

$$Y_1 = X_1^{\frac{1}{\alpha}} - hY_2^{\beta}$$

$$Y_2 = X_2^{\frac{1}{\alpha}}$$

where $\alpha > 1$. Then maximizing profits firm 2 will choose

$$X_2 = \left(\frac{p}{\alpha w} \right)^{\frac{\alpha}{\alpha-1}}$$

and

$$Y_2 = \left(\frac{p}{\alpha w} \right)^{\frac{1}{\alpha-1}}$$

Firm 1 will choose

$$X_1 = 0$$

or

$$X_1 = \left(\frac{p}{aw}\right)^{\frac{a}{a-1}}$$

depending on whether or not it can earn non-negative profits. We can write firm 1's profits, Π_1 , as a function of p/w .

Since

$$\begin{aligned} Y_1 &= \left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h\left(\frac{p}{aw}\right)^{\frac{\beta}{a-1}} \\ \Pi_1 &= p \left[\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h\left(\frac{p}{aw}\right)^{\frac{\beta}{a-1}} \right] - w \left(\frac{p}{aw}\right)^{\frac{a}{a-1}} \\ &= \left(\frac{p}{aw}\right)^{\frac{1}{a-1}} \left[p - ph\left(\frac{p}{aw}\right)^{\frac{\beta-1}{a-1}} - \frac{p}{a} \right] \end{aligned}$$

Since $a > 1$, this expression is positive if

$$\left(1 - \frac{1}{a}\right) > h\left(\frac{p}{aw}\right)^{\frac{\beta-1}{a-1}}$$

Three cases, which differ in terms of the change in marginal damage when Y_2 increases, can be distinguished. Marginal damage, defined as

$\frac{\partial F_1}{\partial Y_2}$, is increasing if $\frac{\partial^2 F_1}{\partial Y_2^2} > 0$; constant if $\frac{\partial^2 F_1}{\partial Y_2^2} = 0$, and decreasing

if $\frac{\partial^2 F_1}{\partial Y_2^2} < 0$.

Case 1: If marginal damage is constant, $\beta = 1$ and the sign of profit is independent of p/w , depending only on whether $1 - 1/a > h$, in which case profits are always positive, or $1 - 1/a < h$, in which case profits are always negative. Since firm 2 produces non-zero output for all $p/w \neq 0$, $1 - 1/a > h$ implies that firm 1 can always produce some output. The supply function then is

$$\begin{aligned} Y_1 + Y_2 &= 2\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} \\ &= (2 - h)\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} \end{aligned}$$

If $1 - 1/a > h$, the supply function is

$$Y_1 + Y_2 = Y_2 = \left(\frac{p}{aw}\right)^{\frac{1}{a-1}}$$

Both functions are continuous, and no problems can arise.

Case 2: If marginal damage is increasing, then $\beta > 1$ and $h\left(\frac{p}{aw}\right)^{\frac{\beta-1}{a-1}}$ is an increasing function of p/w . It is always possible to choose p/w small enough that

$$\left(1 - \frac{1}{a}\right) > h\left(\frac{p}{aw}\right)^{\frac{\beta-1}{a-1}}$$

Therefore for low values of p/w firm 1 can earn positive profits.

As p/w increases, however, we can choose a large enough value to make

$$\left(1 - \frac{1}{a}\right) < h\left(\frac{p}{aw}\right)^{\frac{\beta-1}{a-1}}$$

so that profits become negative. Let R be that value of p/w for which

$$1 - \frac{1}{a} = h\left(\frac{p}{aw}\right)^{\frac{\beta-1}{a-1}}$$

Then for $p/w \leq R$, total supply is

$$Y_1 + Y_2 = 2\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h\left(\frac{p}{aw}\right)^{\frac{\beta}{a-1}} \quad (14)$$

For $p/w \geq R$,

$$Y_1 + Y_2 = Y_2 = \left(\frac{p}{aw}\right)^{\frac{1}{a-1}} \quad (15)$$

Solving for R in

$$1 - \frac{1}{a} = h\left(\frac{1}{aR}\right)^{\frac{\beta-1}{a-1}}$$

gives

$$\frac{1}{aR} = \left(\frac{a-1}{ah}\right)^{\frac{a-1}{\beta-1}}$$

Substituting R for p/w in (14) gives

$$\begin{aligned} Y_1 + Y_2 &= 2\left(\frac{a-1}{ah}\right)^{\frac{1}{\beta-1}} - h\left(\frac{a-1}{ah}\right)^{\frac{\beta}{\beta-1}} \\ &= \left(1 + \frac{1}{a}\right)\left(\frac{a-1}{ah}\right)^{\frac{1}{\beta-1}} \end{aligned} \quad (16)$$

and in (15) gives

$$Y_1 + Y_2 = \left(\frac{a-1}{ah}\right)^{\frac{1}{\beta-1}} \quad (17)$$

Since (16) > (17), the total supply function will jump at R , since at R (14), which defines the supply function up to R , is strictly greater than (15) which defines the supply function from R on. This is illustrated in Figure 5.

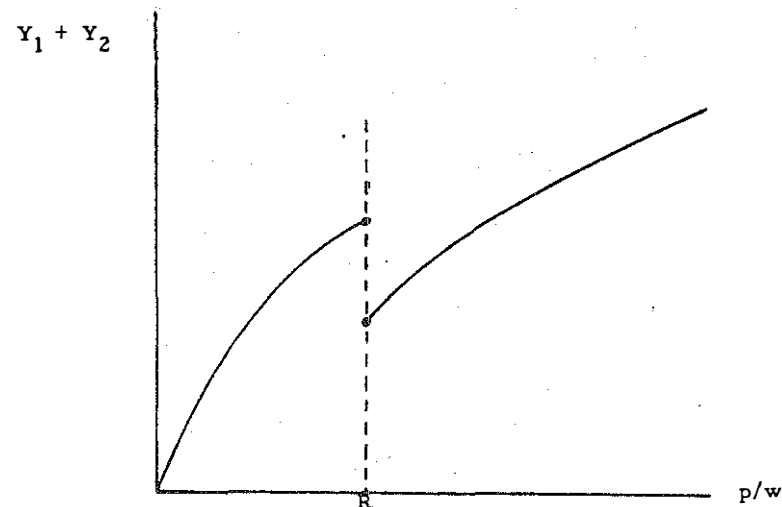


Figure 5

Case 3: If marginal damage is decreasing, then $\beta < 1$ and $h\left(\frac{p}{aw}\right)^{\frac{\beta-1}{\alpha-1}}$ is a decreasing function of p/w . By a train of reasoning identical to that used in Case 2 we can establish that for $p/w < R$ firm 1 cannot earn positive profits; for $p/w > R$ it can. Therefore the supply function is (15) for $0 < p/w \leq R$ and (14) for $p/w > R$. We have established that (14) $>$ (15) at R . Therefore the supply function jumps, as in Figure 6.

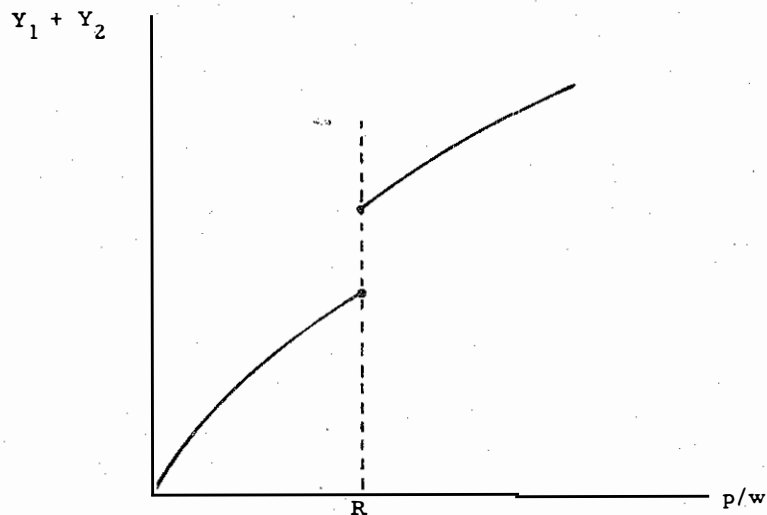


Figure 6

These discontinuities can interfere with the existence of equilibrium. If the demand curve for $Y_1 + Y_2$ goes through the discontinuity, it will never be possible to achieve exact equality between supply and demand. This is more likely in Case 3, of course, since in Case 2 a demand curve must have a positive slope to pass through the discontinuity.

Such a discontinuity does not necessarily preclude the possibility of proving the existence of equilibrium, although proving existence becomes more complex. The standard argument (Arrow and Hahn, p. 169-171, Rothenberg [1960])

is that as more firms and consumers are introduced into the economy it becomes possible to choose various combinations of firms producing zero output and firms producing positive output to approximate demand at the price for which the discontinuity exists. Such an argument cannot necessarily be made in this case. As Rothenberg [1960] pointed out there is no guarantee that non-convexities arising from externalities will vanish as the number of agents in the economy increases. The discontinuity caused by one-way, separable externality will change in different ways depending on how the economy is expanded. If we let the number of polluting firms increase while holding constant the number of sufferers, the discontinuity will vanish since the firms affected by the non-convexity become small relative to the economy. If both classes are increased at the same rate, the size of the discontinuity relative to aggregate supply may remain roughly constant.

The analysis of cases in which there are mutual externalities and decreasing returns is much more complex. To define the total supply correspondence we must make sure that the supply response of any firm to given prices is profit-maximizing with respect not only to the prices but also with respect to the decision of the other firm. When this is done it appears that for some production functions it is impossible for more than one firm to earn positive profits at any prices; in some cases two firms can earn positive profits at all prices; in others more firms can. Whenever market structure is thus independent of prices, the system is well behaved and the aggregate supply correspondence exhibits no discontinuities. We have shown that when market structure changes with price discontinuities can appear.

Again, we turn to a simple example to illustrate some important relationships. Assume two identical firms $i = 1, 2$, with production functions

$$Y_i = X_i^{\frac{1}{\alpha}} - hY_j$$

At any price ratio p/w the input choice of firm i will be either

$$X_i = \left(\frac{p}{aw}\right)^{\frac{a}{a-1}}$$

or $X_i = 0$, depending on whether Y_j is such that firm i can or cannot earn non-negative profits.

We begin by assuming that both firms are operating at interior maxima. We know the input of X_i ; but output Y_i depends on the output of firm j , which in turn depends on Y_i . We cut through this chain by solving

$$Y_i = \left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h \left[\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h Y_i \right] \quad (17)$$

Therefore

$$Y_i = \frac{1}{1+h} \left(\frac{p}{aw}\right)^{\frac{1}{a-1}}$$

and

$$\Pi_i = p \left(\frac{p}{w}\right)^{\frac{1}{a-1}} \left[\frac{1}{1+h} \frac{1}{a^{1-a}} - \frac{a}{a^{1-a}} \right]$$

Profits Π_i will be positive, with both firms in operation, for all prices when h is less than some critical value. We solve

$$\frac{1}{1+h} - \frac{a}{a^{1-a}} = 0$$

to find this critical value, which is $h = a - 1$. Since a is a measure

of how strongly returns to scale decrease, it is a relation between the magnitude of the externality and returns to scale which determines how many firms can operate in equilibrium. Since the possibility of earning non-negative profits is independent of p/w , the economy will always have two firms operating if $h < a - 1$, and one firm if $h > a - 1$.

This suggests that if a is sufficiently large, it may be possible to have more than two firms in operation. Suppose there are $n + 1$ identical firms, with production functions

$$Y_i = X_i^{\frac{1}{a}} - h \sum_{j \neq i} Y_j$$

Clearly every firm which is in operation will choose identical X_i . Thus we can again solve

$$\begin{aligned} Y_i &= \left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h \sum_{j \neq i} \left[\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h n Y_1 \right] \\ &= \left(\frac{p}{aw}\right)^{\frac{1}{a-1}} + h n \left[\left(\frac{p}{aw}\right)^{\frac{1}{a-1}} - h n Y_1 \right] \end{aligned} \quad (18)$$

where the substitution $n Y_1 = \sum_{j \neq i} Y_j$ follows from the fact that all firms in operation will produce identical output. Since (18) is identical to (17) except that h is replaced by $h n$, it follows that

$$\Pi_i > 0 \quad \text{for all } i \text{ if } h n < a - 1$$

For example, if h is just less than 1 and $a = 3$, $n + 1 = 3$ firms can earn positive profits simultaneously.

5. CONCLUSION

The characterization of production sets implicit in some classic partial equilibrium analyses of externalities has been shown to differ significantly from the minimal assumption used in proving the existence of general equilibrium of a competitive economy with externalities. Since Davis and Whinston [1962] numerous papers have appeared deriving properties of an economic system with "separable externalities." Separable externalities are mathematically quite tractable, and often make it possible to devise institutions which can achieve Pareto Optimality in the presence of externalities. Within the class of production functions which give rise to "separable externalities" are, however, production functions which violate the convexity properties used in proving existence of equilibrium. [Arrow and Hahn, 1973]

This non-convexity has implications for market structure and the existence of equilibrium. We have shown that when both the cost function and the production function are separable, the production function must be homogeneous of degree one in the variables under the control of the firm. In this case only one firm affected by externalities can exist in competitive equilibrium. It follows that it is uninteresting to examine some hypothetical partial equilibrium system in which two firms are related by externalities separable in both the cost and the production function.

When all firms have identical production functions which are separable, and exhibit decreasing returns to private inputs, it has been shown that the number of firms which can exist in equilibrium is a function of the degree of decreasing returns. When all firms are identical, the number which can simultaneously earn positive profits is independent of prices or, consequently, demand conditions. When firms are dissimilar, as in the case where only one of two

firms is affected by externality, then the number of firms which can earn positive profits may vary with prices. When market structure thus changes with prices, discontinuity can arise in the aggregate supply correspondence. Although this discontinuity is a well-known consequence of non-convexity, it does not always arise as a result of the non-convexity caused by separable externalities. It is a special case because the way in which firms adjust their production plans in the presence of externality can itself serve to eliminate or limit the size of the non-convexity. In many but not all cases this adjustment is sufficient to guarantee the existence of equilibrium.

APPENDIX: ALTERNATIVE DEFINITIONS OF SEPARABILITY

The concept of separability which appears in analysis of externalities differ from concepts of separable utility and production functions found in classical microeconomic theory and from the definition of separable functions used in mathematical programming. We have defined separability in terms which imply that all cross-partial derivatives between Y_2 and variables controlled by firm 1 vanish. The survey of the place of separability in economic theory by Geary and Morishima [1973] contains only one definition of separability equivalent to that used in this paper. It is Frisch's idea of want independence: "the marginal utility of good i depends only on good i ." (Morishima [1973], p.103) Want independence is equivalent to the assumption that relevant cross-partial derivatives vanish. Geary and Morishima comment that "want independence is not invariant under a monotonic transformation . . . the concept is unnecessarily cumbersome; this is underlined by the fact that all of Frisch's results can be obtained from Sono's independence which is free of cardinal concepts." (p.105) Sono's independence characterizes a function which can be written in the form

$$f(X) = F[f^1(X^1) + f^2(X^2, \dots, X^n)]$$

As we have seen, not all production functions which can be written in this form are separable, since the relevant cross-partial of $f(x)$ vanishes only if $F'' = 0$, and not all such functions generate separable cost functions. The deeper difference between Sono's separability and the concept of separability relevant to externalities is that the theorems which have been proved in the literature on separable externalities do require that

$$\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = 0. \text{ An affine transformation of a cost function with this property}$$

generates different supply and demand curves and observably different behavior of the firm. If, for example, $C = Y_1 - Y_2$ and $F(C) = C^2$, $F' = Y_1^2 - 2Y_1Y_2 + Y_2^2$. A firm with this cost function differs in every way from a firm with the cost function $Y_1 + Y_2$.

Another definition of separability related but not identical to that used here is "homogeneous separability." (Morishima, p.103) A function is homogeneously separable if it can be written as

$$U(X) = U[f_1(X_1, \dots, X_r), f_2(X_{r+1}, \dots, X_{r+s}), \dots, f_n(X_{t+1}, \dots, X_u)]$$

where the f_i are homogeneous of degree one. Again, this definition differs from ours in that it is invariant to affine transformation. Moreover, the functional form, although similar to (11), differs in two crucial respects: $h(Y_2)$ need not be homogeneous of degree one, and a homogeneously separable function does not in general satisfy (12). It follows from these comparisons that the results which follow from the definitions of separability used in this paper are somewhat more than special cases of known results.

Davis and Whinston [1962] and Marchand and Russell [1973] refer to a paper by Charnes and Lemke [1954] as the source of their definition of separability. Charnes and Lemke state a definition relevant to mathematical programming which is not in fact identical to that need in analyzing externalities. A function $F(X_1, \dots, X_n)$ is separable under this definition if it is possible to find n functions $g_i(X_1, \dots, X_n)$ such that

$$F(g_1, \dots, g_n) = h_1(g_1) + \dots + h_n(g_n) \quad [\text{Hadley 1964}]$$

Not all functions which are separable in this sense exhibit separable externalities. Consider the production function $F(X_1, \dots, X_n, Y_2)$. The externality is separable only if there are functions g_1 and g_2 such that

$$F(g_1, \dots, g_n, g_{n+1}) = \sum_i h_i[g_i(X_1, \dots, X_n)] + h_{n+1}[g_{n+1}(Y_2)]$$

If the functions $h_i(g_i)$ are not all independent of Y_2 then making the function appear additive fails to separate the externality. The cost function makes things even clearer. If $C = C_1[g_1(Y_1, Y_2)] + C_2[g_2(Y_1, Y_2)]$, there is no reason for

$$\frac{\partial^2 C}{\partial Y_1 \partial Y_2} = \sum_{i=1}^2 \left[C_i \frac{\partial g_i}{\partial Y_1} \frac{\partial g_i}{\partial Y_2} + C_i \frac{\partial^2 g_i}{\partial Y_1 \partial Y_2} \right]$$

to equal zero. Thus the programming definition of separability fails to characterize separable externalities.

To summarize: theorems on separable externalities depend on the assumption that $\frac{\partial^2 C}{\partial Y_1 \partial Y_2}$ vanish everywhere. The only functions with this property are of the form $C_1(Y_1) + C_2(Y_2)$. The class of production functions which give rise to such cost functions is not completely characterized by any of the common definitions of separability found elsewhere in economics or programming.

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